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Superfluid plasmas: Multivelocity nonlinear hydrodynamics of superfluid solutions with charged condensates coupled electromagnetically

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Four levels of nonlinear hydrodynamic description are presented for a nondissipative multicondensate solution of superfluids with vorticity. First, the multivelocity superfluid (MVSF) theory is extended to the case of a multivelocity superfluid plasma (MVSP), in which some of the superfluid condensates (protons, say) are charged and coupled electromagnetically to an additional, normal, charged fluid (electrons). The resulting drag-current density is derived due to the electromagnetic coupling of the condensates with the normal fluids. For the case of one charged condensate, the MVSP equations simplify to what we call superfluid Hall magnetohydrodynamics (SHMHD) in the approximation that displacement current and electron inertia are negligible, and local charge neutrality is imposed. The contribution of the charged condensate to the Hall drift force is determined. In turn, neglecting the Hall effect in SHMHD gives the equations of superfluid magnetohydrodynamics (SMHD). Each set of equations (MVSF, MVSP, SHMHD, and SMHD) is shown to be Hamiltonian and to possess a Poisson bracket associated with the dual space of a corresponding semidirect-product Lie algebra with a generalized two-cocycle defined on it. Topological conservation laws (helicities) associated with the kernels of these Lie algebras are also discussed as well as those associated physically with generalized Kelvin theorems for conservation of superfluid circulation around closed loops moving with the *normal* fluid.

INTRODUCTION

In a solution of ^{3}He atoms in liquid ^{4}He at a sufficiently low temperature, a phase transition to superfluidity can occur via Cooper pairing of the ^{3}He Fermi particles. At temperatures below the phase-transition point, two types of condensate and, consequently, two types of superfluid will exist simultaneously. The hydrodynamics of such a solution can be described by a three-fluid model, with two superfluid velocities and one normal-fluid velocity. These equations have been derived by Andreev and Bashkin,[1] with allowance for dragging of the ^{3}He component by the ^{4}He flow. The three-fluid hydrodynamic equations have also been studied by Khalatnikov,[2] Galasiewicz,[3] Mineev,[4] and Volovik, Mineev, and Khalatnikov.[5] More exotic systems having two superfluid condensates may also exist, e.g., in the "prephase" of neutron stars (Ambartsumyan and Saakyan[6]) in which Cooper pairs are formed as a result of hadronic interactions at high density (Ginzburg[7]).

Another example of such a system with two condensates consists of a solution of protons and neutrons distributed as impurities in a heavy-metal matrix (Lowy and Woo[8]). Such a system has the properties of a quantum crystal, due to the large mass difference between the impurity particles and the matrix atoms (Andreev and Lifshitz[9]). Under certain conditions, the neutron-proton impurities can undergo a phase transition to a superconducting state, again via formation of Cooper pairs (Gelikman,[10] referenced in Vardanyan and Sedrakyan[11]).

In this paper we first discuss the Hamiltonian structure of the ideal multifluid hydrodynamic equations for multivelocity superfluid (MVSF) theory. Then, reasoning based on this Hamiltonian structure allows us to extend the MVSF equations to a multivelocity plasma (MVSP) model, by allowing some of the condensates to be charged (as in the case of neutron and proton condensates) and interact electromagnetically with an additional charged normal component (e.g., electrons). Next, we introduce a superfluid Hall magnetohydrodynamic approximation (SHMHD, including the Hall electric field induced by currents flowing transversely to the local magnetic field). This approximation applies to the multivelocity superfluid plasma system in the case of only one charged condensate in a neutralizing background normal fluid. This is derived by neglecting displacement current and electron inertia in the MVSP equations, and imposing local charge neutrality (which is dynamically preserved). The MVSP analysis yields the drag-current density [see Eq. (24)] due to the coupling of the condensates with the normal fluids, as well as the dynamics of the neutron and proton vortices. The SHMHD analysis reveals the contribution of the charged condensate to the Hall field [see Eq. (34)]. Neglecting the Hall field in the superfluid Hall magnetohydrodynamics (SHMHD) equations finally results in the equations of superfluid magnetohydrodynamics (SMHD).

For each of these four Hamiltonian systems (MVSF, MVSP, SHMHD, SMHD), additional conservation laws exist due to the noncanonical nature of their Poisson brackets. These Poisson brackets are associated in the text to the dual spaces of Lie algebras of semidirect-product type. The nontrivial kernels of these Lie algebras correspond to conservation laws existing for *any* Hamiltonian that may be expressed in the space of physical variables for each system. These additional conservation laws are helicities (topological winding numbers of superfluid vortex lines) that derive from generalized Kelvin theorems describing conservation of superfluid circulation around any closed loop that moves with the normal fluid.

MULTIVELOCITY SUPERFLUIDS

The equations of multivelocity hydrodynamics of superfluid solutions in the absence of dissipation are given in the following form (cf. Andreev and Bashkin,[1] who consider the case of two condensates, labeled by $\alpha=1,2$),

$$\partial_t\rho_\alpha=-\operatorname{div}(\rho_\alpha\mathbf{v}_n+\mathbf{P}_\alpha),\quad \alpha=1,2,\ldots,m \tag{1a}$$

$$\partial_t P_i=-\partial_k\pi_i^k,\quad i,k=1,2,\ldots,N \tag{1b}$$

$$\partial_t S=-\operatorname{div}(S\mathbf{v}_n)\ , \tag{1c}$$

$$\partial_t\mathbf{v}^\alpha=-\nabla(\mu^\alpha-\tfrac{1}{2}v_n^2+\mathbf{v}_n\cdot\mathbf{v}^\alpha)+\mathbf{v}_n\times\operatorname{curl}\mathbf{v}^\alpha\ , \tag{1d}$$

where the three-dimensional vector notation in Eq. (1) is expressible in N-dimensional component notation as, e.g.,

$$(\mathbf{v}_n\times\operatorname{curl}\mathbf{v}^\alpha)_i=v_n^j(v_{j,i}^\alpha-v_{i,j}^\alpha)\quad\text{for } i,j=1,2,\ldots,N\ .$$

In (1a)–(1d) the quantities ρ_α, $\alpha=1,2,\ldots,m$, are the mass densities of condensate particles of each kind. The total mass density ρ of the condensates in solution is equal to the sum over species, $\rho=\sum_\alpha\rho_\alpha$. The quantities $\mathbf{v}_n$ and $\mathbf{v}^\alpha$, $\alpha=1,2,\ldots,m$ are, respectively, the velocities of the normal flow and the m superfluid flows. The total momentum density is denoted

$$\mathbf{P}=\rho\mathbf{v}_n+\sum_\alpha\mathbf{P}_\alpha$$

in (1b), and S is the entropy density in (1c). Equation (1d) reduces to the usual single-fluid motion equation when $\mathbf{v}^\alpha=\mathbf{v}_n$, and all the fluids move together.

The interpretations of the other quantities in (1) $\mu^\alpha,\mathbf{P}_\alpha$, and the temperature T may be obtained from the following thermodynamic derivative identity (first law of thermodynamics) for the internal energy density of the solution ε in the Galilean frame in which $\mathbf{v}_n=\mathbf{0}$:

$$d\varepsilon=T\,dS+\sum_\alpha[\mu^\alpha d\rho_\alpha+\mathbf{P}_\alpha\cdot d(\mathbf{v}^\alpha-\mathbf{v}_n)]\ . \tag{2}$$

The total energy density in the laboratory frame is equal to

$$E=\tfrac{1}{2}\rho v_n^2+\left[\sum_\alpha\mathbf{P}_\alpha\right]\cdot\mathbf{v}_n+\varepsilon \tag{3a}$$

$$=-\tfrac{1}{2}\rho v_n^2+\mathbf{P}\cdot\mathbf{v}_n+\varepsilon\ . \tag{3b}$$

The momentum flux tensor π_i^k in (1b) has the form

$$\pi_i^k=p\delta_i^k+\sum_\beta(v_i^\beta P_\beta^k)+P_iv_n^k\ , \tag{4}$$

where the pressure p is defined by the Euler relation,

$$p=-\varepsilon+\sum_\alpha(\mu^\alpha\rho_\alpha)+TS\ . \tag{5}$$

Equations (1a)–(1d) comprise a Hamiltonian system, which can be written in the form $\partial_tF=\{H,F\},F\in\{\rho_\alpha,\mathbf{v}^\alpha,S,P_i\}$, with Hamiltonian H given by

$$H=\int d^Nx(-\tfrac{1}{2}\rho v_n^2+\mathbf{P}\cdot\mathbf{v}_n+\varepsilon)\ , \tag{6}$$

where the integrand is obtained from (3b) and N is the dimension of space, with volume element d^Nx. (Physically, $N=3$; however, we prefer to keep N arbitrary for mathematical convenience and transparency of expressions in component notation.) The variational derivatives of H in (6) are found upon using (2) to be

$$\delta H=\int d^Nx\left\{\sum_\alpha(\mu^\alpha-\tfrac{1}{2}v_n^2)\delta\rho_\alpha+T\,\delta S+\sum_\alpha\mathbf{P}_\alpha\cdot\delta\mathbf{v}^\alpha+\mathbf{v}_n\cdot\delta\mathbf{P}+\left[\mathbf{P}-\left[\sum_\alpha\mathbf{P}_\alpha\right]-\rho\mathbf{v}_n\right]\cdot\delta\mathbf{v}_n\right\}\ . \tag{7}$$

The Poisson bracket $\{\ ,\ \}$ in terms of which Eqs. (1a)–(1d) are expressible in Hamiltonian form is given by (summing on repeated indices)

$$\{H,F\}=-\int d^Nx\left[\frac{\delta F}{\delta P_i}\left[(P_k\partial_i+\partial_kP_i)\frac{\partial H}{\delta P_k}+S\partial_i\frac{\delta H}{\delta S}+\rho_\beta\partial_i\frac{\delta H}{\delta\rho_\beta}+(\partial_kv_i^\beta-v_{k,i}^\beta)\frac{\delta H}{\delta v_k^\beta}\right]\right.$$

$$+\left[\frac{\delta F}{\delta S}\partial_kS+\frac{\delta F}{\delta\rho_\alpha}\partial_k\rho_\alpha+\frac{\delta F}{\delta v_i^\alpha}(v_k^\alpha\partial_i+v_{i,k}^\alpha)\right]\frac{\delta H}{\delta P_k} \tag{8a}$$

$$\left.+\left[\frac{\delta F}{\delta\rho_\alpha}\delta_\alpha^\beta\partial_k\frac{\delta H}{\delta v_k^\beta}+\frac{\delta F}{\delta v_i^\alpha}\delta_\beta^\alpha\partial_i\frac{\delta H}{\delta\rho_\beta}\right]\right]\ . \tag{8b}$$

Equations (1) for MVSF result immediately in Hamiltonian form $\partial_tF=\{H,F\}$ by substituting the variational derivatives from (7) into the Poisson bracket (8).

The Poisson bracket (8) for MVSF is the sum of two parts: a semidirect-product piece and a generalized two-cocycle. The first part (8a) represents the natural Poisson bracket (see Holm and Kupershmidt[12] and Kupershmidt[13]) on the dual space associated with the Lie algebra

$$L_1 = D \circledS [\Lambda^0 \oplus (\Lambda^0_\alpha \underset{\alpha}{\oplus} \Lambda^{N-1}_\alpha)] . \tag{9}$$

The symbol Ⓢ denotes the semidirect product with respect to the natural action of vector fields D on differential k-forms Λ^k and $\oplus_\alpha$ denotes direct sum over α. The corresponding commutator for this Lie algebra is given by Holm and Kupershmidt,[12]

$$[(X;f;g_\alpha;\theta^\alpha),(\bar{X};\bar{f};\bar{g}_\alpha;\bar{\theta}^\alpha)] = ([X,\bar{X}];X(\bar{f})-\bar{X}(f);X(\bar{g}_\alpha)-\bar{X}(g_\alpha);X(\bar{\theta}^\alpha)-\bar{X}(\theta^\alpha)) . \tag{10}$$

Dual coordinates on the Lie algebra L_1 in (9) are $\mathbf{P}$ dual to $X \in D$; S to $f \in \Lambda^0$; ρ_α to $g_\alpha \in \Lambda^0_\alpha$; and $\mathbf{v}^\alpha$ to $\theta^\alpha \in \Lambda^{N-1}_\alpha$. The second part of the MVSF bracket (8b) (the $\rho_\alpha - \mathbf{v}^\alpha$ piece) represents the generalized two-cocycle on L_1

$$\omega_1((X;f;g_\alpha;\theta^\alpha),(\bar{X};\bar{f};\bar{g}_\alpha;\bar{\theta}^\alpha)) = g_\alpha \bar{\theta}^\alpha_{k,k} + \theta^\alpha_k \bar{g}_{\alpha,k} . \tag{11}$$

Indeed, ω_1 is obviously skew symmetric and satisfies

$$\omega_1([(X;\cdots),(\bar{X};\cdots)],(\bar{\bar{X}};\cdots)]) + \text{c.p.} \sim 0 , \tag{12}$$

where c.p. stands for cyclic permutation of the unadorned quantities in (12) with those having a single overbar and a double overbar, the ellipses refer to the elements acted upon by vector fields X, $\bar{X}$, and $\bar{\bar{X}}$, and the equivalence relation $a \sim b$ means

$$(a-b) \in \sum_k \mathrm{Im}\,\partial/\partial x^k$$

(see Kupershmidt,[13] Chap. viii). To show that (12) holds, use (10) to express the left-hand side of (12) as

$$\begin{aligned}
&(X_i \bar{g}_{\alpha,i} - \bar{X}_i g_{\alpha,i}) \bar{\bar{\theta}}^\alpha_{k,k} + [(X_i \bar{\theta}^\alpha_k)_{,i} - \bar{\theta}^\alpha_s X_{k,s} - (\bar{X}_i \theta^\alpha_k)_{,i} + \theta^\alpha_s \bar{X}_{k,s}] \bar{\bar{g}}_{\alpha,k} + \text{c.p.} \\
&\sim (X_i \bar{g}_{\alpha,i} \bar{\bar{\theta}}^\alpha_{k,k} + \text{c.p.}) - (X_i \bar{\bar{g}}_{\alpha,i} \bar{\theta}^\alpha_{k,k} + \text{c.p.}) - (X_i \bar{\theta}^\alpha_k \bar{\bar{g}}_{\alpha,ki} + \text{c.p.}) \\
&\quad + [X_i (\bar{\theta}^\alpha_s \bar{\bar{g}}_{\alpha,i})_{,s} + \text{c.p.}] + (X_i \bar{\bar{\theta}}^\alpha_k \bar{g}_{\alpha,ki} + \text{c.p.}) - [X_i (\bar{\bar{\theta}}^\alpha_s \bar{g}_{\alpha,i})_{,s} + \text{c.p.}] = 0 .
\end{aligned} \tag{13}$$

Remark A. For any α, if curl $\mathbf{v}^\alpha = \mathbf{0}$ initially, it remains so. Thus, the potential flows $\mathbf{v}^\alpha = \nabla \phi^\alpha$ form an invariant subsystem of (1a)–(1d). For this subsystem, the Poisson bracket (8) becomes (in the case when curl $\mathbf{v}^\alpha = \mathbf{0}$ for *all* α)

$$\begin{aligned}
\{H,F\} = -\int d^N x \Bigg[&\frac{\delta F}{\delta P_i} \left[(P_k \partial_i + \partial_k P_i) \frac{\delta H}{\delta P_k} + S \partial_i \frac{\delta H}{\delta S} + \rho_\beta \partial_i \frac{\delta H}{\delta \rho_\beta} - \phi^\alpha_{,i} \frac{\delta H}{\delta \phi^\alpha} \right] \\
&+ \left[\frac{\delta F}{\delta S} \partial_k S + \frac{\delta F}{\delta \rho_\beta} \partial_k \rho_\beta + \frac{\delta F}{\delta \phi^\alpha} \phi^\alpha_{,k} \right] \frac{\delta H}{\delta P_k} && \text{(8a}')\\
&+ \left[\frac{\delta F}{\delta \phi^\alpha} \frac{\delta H}{\delta \rho_\alpha} - \frac{\delta H}{\delta \phi^\alpha} \frac{\delta F}{\delta \rho_\alpha} \right] \Bigg] . && \text{(8b}')
\end{aligned}$$

The Poisson bracket (8′) is the sum of a semidirect-product piece (8a′) and a symplectic two-cocycle piece (8b′). The semidirect-product piece is associated with the dual of the Lie algebra [cf. (9)]

$$L_1' = D \circledS [\Lambda^0 \oplus (\Lambda^0_\alpha \underset{\alpha}{\oplus} \Lambda^N_\alpha)] . \tag{9$'$}$$

Dual coordinates on the Lie algebra L_1' in (9′) are the same as for the Lie algebra L_1 in (9), except that the superfluid velocity potential ϕ^α is dual to the elements Λ^N_α (densities). Notice that the formula $\mathbf{v}^\alpha = \nabla \phi^\alpha$ is dual to the homomorphism for Lie algebras

$$\mathrm{Id} \circledS [\mathrm{Id} \oplus (\mathrm{Id}_\alpha \underset{\alpha}{\oplus} d_\alpha)]: \; L_1 \to L_1' ,$$

where Id denotes the identity and d denotes exterior differentiation.

Remark B. The above interpretations of dual coordinates are physically natural: ρ and S are densities, and so are dual to 0-forms Λ^0; v^α_i are components of the circulation 1 forms $v^\alpha = v^\alpha_i dx^i$, which are dual to $(N-1)$ forms, Λ^{N-1}; and $\mathbf{P}$ is a momentum per unit volume (1-form density) and, so, is dual to vector fields. See, e.g., Holm and Kupershmidt,[12,14] Holm, Kupershmidt, and Levermore,[15] Marsden, Ratiu, and Weinstein,[16] and various articles in Marsden[17] for further examples, references, and discussions of the Lie algebraic interpretations of Poisson brackets in ideal hydrodynamics.

MULTIVELOCITY SUPERFLUID PLASMA (MVSP)

Equations (1a)–(1d) and their associated Poisson bracket (8) for the multivelocity superfluid (MVSF) model can be generalized to the case of a multivelocity superfluid plasma (MVSP), in which some of the condensates are charged (e.g., neutrons and protons) and interact electromagnetically with another, additional, charged normal component (the electrons). In this case, four new variables are added: $\mathbf{A}$, the vector potential of the electromagnetic field; $\mathbf{D}$, the electric displacement vector; $\tilde{\rho}$, the electron mass density; and $\tilde{\mathbf{M}}$, the electron momentum density, related to the electron fluid velocity, mass density, and vector potential by minimal coupling, i.e.,

$$\tilde{\mathbf{M}}=\tilde{\rho}(\tilde{\mathbf{v}}+\tilde{a}\,\mathbf{A})\ , \tag{14}$$

where $\tilde{a}$ is the electron charge-to-mass ratio, which is negative. The total momentum density becomes

$$\mathbf{P}=\sum_\alpha \mathbf{P}_\alpha+\rho\mathbf{v}_n+\sum_\alpha a^\alpha\rho_\alpha\mathbf{A}\ , \tag{15}$$

where, as before, $\rho=\sum_\alpha\rho_\alpha$ is the total mass density of condensates, and a^α (without tilde) is the charge-to-mass ratio of species α. (Of course, a^α vanishes for any uncharged species.) Likewise, we define altered circulation components for the condensates,

$$\mathbf{u}^\alpha=\mathbf{v}^\alpha+a^\alpha\mathbf{A}\ , \tag{16}$$

according to the minimal-coupling hypothesis for the charged superfluid species.

The total energy for the MVSP system becomes [cf. Eq. (6)]

$$H=\int d^Nx\left[-\tfrac{1}{2}\rho v_n^2+\left[\mathbf{P}-\sum_\alpha a^\alpha\rho_\alpha\mathbf{A}\right]\cdot\mathbf{v}_n+\varepsilon(S,\{\rho_\alpha\},\{\mathbf{v}^\beta\})+\frac{1}{2\tilde{\rho}}\,|\tilde{\mathbf{M}}-\tilde{a}\tilde{\rho}\mathbf{A}|^2+\tilde{\varepsilon}(\tilde{\rho})+\varepsilon_{EM}(\mathbf{A},\mathbf{D})\right]\ , \tag{17}$$

where the energy densities ε, $\tilde{\varepsilon}$, and ε_{EM} satisfy the identities [cf. Eq. (2)]

$$d\varepsilon=T\,dS+\sum_\alpha(\mu^\alpha d\rho_\alpha)+\sum_\alpha\mathbf{P}_\alpha\cdot(d\mathbf{u}^\alpha-a^\alpha d\,\mathbf{A}-d\mathbf{v}_n)\ ,$$
$$d\tilde{\varepsilon}=\tilde{\rho}^{\,-2}\tilde{p}\,d\tilde{\rho},\quad d\varepsilon_{\mathrm{EM}}=\mathrm{curl}\mathbf{H}\cdot d\,\mathbf{A}+\mathbf{E}\cdot d\mathbf{D}\ , \tag{18}$$

with $\tilde{p}$ the electron pressure, $\mathbf{H}$ the magnetic field intensity, and $\mathbf{E}$ the electric field. In component notation in N dimensions, $\mathrm{curl}\mathbf{H}\cdot d\,\mathbf{A}$ in (18) is given by $H_{ij,j}dA_i$, with H_{ij} on antisymmetric tensor. The variational derivatives of the Hamiltonian H in (17) are obtained with the help of (18) from the expression

$$\delta H=-\int d^Nx\left[\sum_\alpha(\mu^\alpha-\tfrac{1}{2}v_n^2-a^\alpha\mathbf{A}\cdot\mathbf{v}_n)\delta\rho_\alpha+T\,\delta S+\sum_\alpha(\mathbf{P}_\alpha\cdot\delta\mathbf{u}^\alpha)+\mathbf{v}_n\cdot\delta\mathbf{P}\right.$$
$$+\left[\mathbf{P}-\sum_\alpha\mathbf{P}_\alpha-\rho\mathbf{v}_n-\sum_\alpha a^\alpha\rho_\alpha\mathbf{A}\right]\cdot\delta\mathbf{v}_n+\tilde{\mathbf{v}}\cdot\delta\tilde{\mathbf{M}}+[-\tfrac{1}{2}\tilde{v}^{\,2}+\tilde{\varepsilon}'(\tilde{\rho})-\tilde{a}\tilde{\mathbf{v}}\cdot\mathbf{A}]\delta\tilde{\rho}+\mathbf{E}\cdot\delta\mathbf{D}$$
$$\left.+\left[\mathrm{curl}\mathbf{H}-\tilde{a}\tilde{\rho}\tilde{\mathbf{v}}-\sum_\alpha a^\alpha(\mathbf{P}_\alpha+\rho_\alpha\mathbf{v}_n)\right]\cdot\delta\,\mathbf{A}\right]\ . \tag{19}$$

The equations of motion for the superfluid plasma system can now be written as a Hamiltonian system, by extending the Poisson bracket (8) to the form

$$\{H,F\}=-\int d^Nx\left[\frac{\delta F}{\delta P_i}\left[(P_k\partial_i+\partial_kP_i)\frac{\delta H}{\delta P_k}+S\partial_i\frac{\delta H}{\delta S}+\rho_\beta\partial_i\frac{\delta H}{\delta\rho_\beta}+(\partial_ku_i^\beta-u_{k,i}^\beta)\frac{\delta H}{\delta u_k^\beta}\right]\right.$$
$$+\left[\frac{\delta F}{\delta S}\partial_kS+\frac{\delta F}{\delta\rho_\alpha}\partial_k\rho_\alpha+\frac{\delta F}{\delta u_i^\alpha}(u_k^\alpha\partial_i+u_{i,k}^\alpha)\right]\frac{\delta H}{\delta P_k} \tag{20a}$$
$$+\left[\frac{\delta F}{\delta\rho_\alpha}\delta_\alpha^\beta\partial_k\frac{\delta H}{\delta u_k^\beta}+\frac{\delta F}{\delta u_i^\alpha}\delta_\beta^\alpha\partial_i\frac{\delta H}{\delta\rho_\beta}\right] \tag{20b}$$
$$+\frac{\delta F}{\delta\tilde{M}_i}\left[(\tilde{M}_k\partial_i+\partial_k\tilde{M}_i)\frac{\delta H}{\delta\tilde{M}_k}+\tilde{\rho}\partial_i\frac{\delta H}{\delta\tilde{\rho}}\right]+\frac{\delta F}{\delta\tilde{\rho}}\partial_k\tilde{\rho}\frac{\delta H}{\delta\tilde{M}_k} \tag{20c}$$
$$\left.+\left[\frac{\delta F}{\delta A_i}\frac{\delta H}{\delta D^i}-\frac{\delta F}{\delta D^i}\frac{\delta H}{\delta A_i}\right]\right]\ , \tag{20d}$$

whose Lie-algebraic nature will be discussed later in Eq. (26).

On substituting the functional derivatives from (19) into the Poisson bracket (20), we find the following system of equations for the multivelocity superfluid plasma (MVSP):

$$\partial_t\rho_\alpha=-\mathrm{div}(\mathbf{P}_\alpha+\rho_\alpha\mathbf{v}_n),\quad \alpha=1,2,\ldots,m \tag{21a}$$

$$\partial_tS=-\mathrm{div}(S\mathbf{v}_n)\ , \tag{21b}$$

$$\partial_t\mathbf{v}^\alpha=-\boldsymbol{\nabla}(\mu^\alpha-\tfrac{1}{2}v_n^2+\mathbf{v}^\alpha\cdot\mathbf{v}_n)+\mathbf{v}_n\times\mathrm{curl}\mathbf{v}^\alpha+a^\alpha(\mathbf{E}+\mathbf{v}_n\times\mathbf{B})\ , \tag{21c}$$

$$\partial_t \mathbf{A} = -\mathbf{E} \;, \tag{21d}$$

$$\partial_t \mathbf{D} = \mathrm{curl}\mathbf{H} - \tilde{a}\tilde{\rho}\tilde{\mathbf{v}} - \sum_\alpha a^\alpha(\mathbf{P}_\alpha + \rho_\alpha \mathbf{v}_n) \;, \tag{21e}$$

$$\partial_t \tilde{\rho} = -\mathrm{div}(\tilde{\rho}\tilde{\mathbf{v}}) \;, \tag{21f}$$

$$\partial_t \tilde{v}_i = \tilde{a}E_i - \tilde{\rho}^{-1}\tilde{\mathbf{p}}_{,i} - \tilde{v}^k \tilde{v}_{i,k} + \tilde{a}\tilde{v}^k(A_{i,k} - A_{k,i}) \;, \tag{21g}$$

$$\partial_t (P_i + \tilde{M}_i + D^k A_{k,i}) = -\partial_k \pi_i^k \;, \tag{21h}$$

where the momentum flux tensor π_i^k is now given by [writing $\varepsilon_{EM} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$]

$$\pi_i^k = [p + \tilde{p} + \tfrac{1}{2}(\mathbf{E}\cdot\mathbf{D} - \mathbf{B}\cdot\mathbf{H})]\delta_i^k + u_i^\beta P_\beta^k + P_i v_n^k + \tilde{M}_i \tilde{v}^k \;. \tag{22}$$

The dynamical equation for the electrons [obtained using Eqs. (18), (19), and (20c)],

$$\partial_t \tilde{M}_i = -\partial_i \tilde{p} - (\tilde{M}_i \tilde{v}^k)_{,k} - \tilde{M}_k \tilde{v}^k_{\;,i} \;, \tag{23}$$

has been used with definition (14) and Eq. (21d) to obtain the electron velocity equation (21g).

The MVSP equations (21a)–(21c) and (21h) extend the previous MVSF equations (1a)–(1d) by modifying the motion equations (21c) of the charged condensates to include the Lorentz force and modifying the total momentum in (21h) and momentum flux tensor in (22) to include the contributions from the electrons and the electromagnetic field. The other MVSP Eqs. (21f)–(21g) provide the hydrodynamics of the electrons, while Eqs. (21d)–(21e) give the Maxwell dynamics of the electromagnetic fields $\mathbf{D}$ and $\mathbf{A}$, with charge current density given by

$$\mathbf{J} = \tilde{a}\tilde{\rho}\tilde{\mathbf{v}} + \sum_\alpha a^\alpha(\mathbf{P}_\alpha + \rho_\alpha \mathbf{v}_n) \;. \tag{24}$$

The dynamical system (21a)–(21h) preserves the static Maxwell equations, with $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$,

$$\mathrm{div}\mathbf{D} = \tilde{a}\tilde{\rho} + \sum_\alpha a^\alpha \rho_\alpha \;, \tag{25a}$$

$$\mathrm{div}\mathbf{B} = \mathbf{0} \;. \tag{25b}$$

Consequently, Eqs. (25a) and (25b) can be taken as initial conditions.

The Poisson bracket (20) is the sum of four parts: the semidirect-product piece (20a) with generalized two-cocycle (20b) as in (8); another semidirect-product piece (20c) for $\tilde{\mathbf{M}}$ and $\tilde{\rho}$; and the canonical $\mathbf{D}-\mathbf{A}$ bracket (20d). Except for the two-cocycles (20b) and (20d) (the $\rho_\alpha - \mathbf{u}^\alpha$ and $\mathbf{D}-\mathbf{A}$ pieces), the bracket (20) may be associated to the dual of the Lie algebra

$$L_2 = L_1 \oplus D \circledS \Lambda^0 = [D \circledS (\Lambda^0 \oplus \Lambda^0_\alpha \underset{\alpha}{\oplus} \Lambda^{n-1}_\alpha)] \oplus [D \circledS \Lambda^0] \;, \tag{26}$$

where L_1 is given by (9), and dual coordinates on L_1 are the same as before except that $\mathbf{v}^\alpha$ is now named $\mathbf{u}^\alpha$; and on the second summand $D \circledS \Lambda^0$ in (26), dual coordinates are $\tilde{\mathbf{M}}$ dual to D and $\tilde{\rho}$ dual to Λ^0.

SUPERFLUID HALL MAGNETOHYDRODYNAMICS (SHMHD)

A magnetohydrodynamic approximation including Hall current effects can be developed from the MVSP equations (21a)–(21h), as follows. Specializing to the case of only one charged condensate for $\alpha = 1$, we have (with Kronecker delta δ_1^α)

$$a^\alpha = a\delta_1^\alpha \;, \tag{27}$$

in (21c), (21e), (24), and (25a).

Neglecting the electron inertia, we may set $(1/\tilde{a}) = 0$ in (21g) and use (21d) to find, in vector notation,

$$\partial_t \mathbf{A} = \tilde{\mathbf{v}} \times \mathbf{B} \;. \tag{28}$$

Thus, the magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$ in this approximation is frozen into the *electron* fluid. The electron fluid velocity $\tilde{\mathbf{v}}$ is expressible by using (24) for the current density and defining $\tilde{n} = \tilde{a}\tilde{\rho}$ to be the electron charge density as

$$\tilde{\mathbf{v}} = \tilde{n}^{-1}[\mathbf{J} - a(\mathbf{P}_1 + \rho_1 \mathbf{v}_n)] \quad \text{with } \tilde{n} = \tilde{a}\tilde{\rho} \;. \tag{29}$$

Thus (28) implies Ohm's law for a multivelocity superfluid with one charged condensate moving across a magnetic field in an electron background, namely,

$$\partial_t \mathbf{A} = -\mathbf{E} = -\left[\frac{a\rho_1}{\tilde{n}}\right]\mathbf{v}_n \times \mathbf{B} + \tilde{n}^{-1}\mathbf{J}\times\mathbf{B} - a\tilde{n}^{-1}\mathbf{P}_1 \times \mathbf{B} \;. \tag{30}$$

The second term on the right-hand side of (30) is the classical drift field due to the Hall effect, while the last term is an additional, superfluid, drift field due to electromagnetic coupling of the charged condensate with the normal fluids. The diamagnetic drift due to electron pressure gradient has been neglected, since it contributes nothing to the other motion equations for SHMHD.

Neglecting relativistic effects, we now set $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$ and take the limit as $\epsilon \to 0$. Hence (21e) and (25a) become

$$\mathbf{J} = \mu_0^{-1}\mathrm{curl}\mathbf{B} \;, \tag{31}$$

and

$$\tilde{n} + a\rho_1 = 0 \;, \tag{32}$$

respectively. Summing the continuity equation (21a) for $\alpha = 1$ with (21f), and using (29) gives

$$\partial_t (a\rho_1 + \tilde{n}) = 0 \;, \tag{33}$$

so that local charge neutrality, expressed in (32), may be taken as an initial condition that is subsequently preserved. The Lorentz force in (21c) then becomes

$$a(\mathbf{E} + \mathbf{v}_n \times \mathbf{B}) = \rho_1^{-1}\mathbf{J}\times\mathbf{B} - a\rho_1^{-1}\mathbf{P}_1 \times \mathbf{B} \;. \tag{34}$$

The terms on the right-hand side of (34) constitute the Hall drift forces produced by the charged condensate current.

The equations of superfluid Hall magnetohydrodynamics (SHMHD) may now be collected, after using $a\rho_1/\tilde{n}=-1$ and taking the limit $\epsilon\to 0$ in (21h),

$$\partial_t \rho_\alpha = -\mathrm{div}(\rho_\alpha \mathbf{v}_n + \mathbf{P}_\alpha) , \tag{35a}$$

$$\partial_t S = -\mathrm{div}(S\mathbf{v}_n) , \tag{35b}$$

$$\begin{aligned}\partial_t \mathbf{v}^\alpha &= \mathbf{v}_n \times \mathrm{curl}\mathbf{v}^\alpha - \nabla(\mu^\alpha - \tfrac{1}{2}v_n^2 + \mathbf{v}^\alpha\cdot\mathbf{v}_n) \\ &\quad + (\rho_1^{-1}\mathbf{J}\times\mathbf{B} - a\rho_1^{-1}\mathbf{P}_1\times\mathbf{B})\delta_1^\alpha , \end{aligned}\tag{35c}$$

$$\partial_t \mathbf{A} = (\mathbf{v}_n + \rho_1^{-1}\mathbf{P}_1 - a^{-1}\rho_1^{-1}\mathbf{J})\times\mathbf{B} , \tag{35d}$$

$$\partial_t (P_i - a\rho_1 A_i) = -\pi_{i,j}^j , \tag{35e}$$

with

$$\begin{aligned}\pi_i^j &= (p + B^2/2\mu_0)\delta_i^j - \mu_0^{-1}B_iB^j \\ &\quad + (P_i - a\rho_1 A_i)v_n^j + \sum_\beta v_i^\beta P_\beta^j , \end{aligned}\tag{36a}$$

$$\mathbf{B} = \mathrm{curl}\,\mathbf{A}, \quad \mathbf{J} = \mu_0^{-1}\mathrm{curl}\mathbf{B} , \tag{36b}$$

$$\mathbf{P} - a\rho_1 \mathbf{A} = \sum_\alpha \mathbf{P}_\alpha + \rho\mathbf{v}_n . \tag{36c}$$

We now state the main result of this section in the following Proposition.

Proposition. The SHMHD equations (35a)–(35e) form an invariant subsystem of the Hamiltonian system consisting of Hamiltonian [with $B_{ij}=A_{i,j}-A_{j,i}$, and ε satisfying (2)]

$$H = \int d^N x \left[-\tfrac{1}{2}\rho v_n^2 + (\mathbf{P} - a\rho_1\mathbf{A})\cdot\mathbf{v}_n + \varepsilon(S,\{\rho_\alpha\},\{\mathbf{v}^\beta\}) + \frac{1}{4\mu_0}B_{ij}B_{ij} \right] , \tag{37}$$

and Poisson bracket

$$\begin{aligned}\{H,F\} = -\int d^N x \Bigg[&\frac{\delta F}{\delta P_i}\left[(P_k\partial_i + \partial_k P_i)\frac{\delta H}{\delta P_k} + S\partial_i\frac{\delta H}{\delta S} + \rho_\beta\partial_i\frac{\delta H}{\delta \rho_\beta} + (\partial_k u_i^\beta - u_{k,i}^\beta)\frac{\delta H}{\delta u_k^\beta} \right] \\ &+ \left[\frac{\delta F}{\delta S}\partial_k S + \frac{\delta F}{\delta \rho_\beta}\partial_k\rho_\beta + \frac{\delta F}{\delta u_i^\alpha}(u_k^\alpha\partial_i + u_{i,k}^\alpha) \right]\frac{\delta H}{\delta P_k} \end{aligned}\tag{38a}$$

$$+ \left[\frac{\delta F}{\delta \rho_\alpha}\delta_\alpha^\beta\partial_k\frac{\delta H}{\delta u_k^\beta} + \frac{\delta F}{\delta u_i^\alpha}\delta_\beta^\alpha\partial_i\frac{\delta H}{\delta\rho_\beta} \right] \tag{38b}$$

$$+ \frac{\delta F}{\delta A_i}\left[\partial_i\frac{\delta H}{\delta\tilde{n}} + \tilde{n}^{-1}(A_{i,k} - A_{k,i})\frac{\delta H}{\delta A_k} \right] + \frac{\delta F}{\delta\tilde{n}}\partial_k\frac{\delta H}{\delta A_k} \Bigg] . \tag{38c}$$

Proof. This proposition may be proven by first substituting the following easily verified expression for the variational derivatives of H in (37),

$$\begin{aligned}\delta H = \int d^N x [&(\mu^\alpha - \tfrac{1}{2}v_n^2 - a\delta_1^\alpha\mathbf{A}\cdot\mathbf{v}_n)\delta\rho_\alpha + T\,\delta S + \sum_\alpha \mathbf{P}_\alpha\cdot\delta\mathbf{u}^\alpha + \mathbf{v}_n\cdot\delta\mathbf{P} \\ &+ (\mu_0^{-1}\mathrm{curl}\mathbf{B} - a\mathbf{P}_1 - a\rho_1\mathbf{v}_n)\cdot\delta\mathbf{A} + (\mathbf{P} - \sum_\alpha\mathbf{P}_\alpha - \rho\mathbf{v}_n - a\rho_1\mathbf{A})\cdot\delta\mathbf{v}_n] , \end{aligned}\tag{39}$$

into the Poisson bracket (38). Next, combining the resulting Hamiltonian equations for ρ_1 and $\tilde{n}$ yields preservation of the local neutrality relation (32) via (33). Eliminating $\tilde{n}$ from these Hamiltonian equations finally recovers the HMHD system (35a)–(35e). This calculation proves the Proposition.

Remark. The Poisson bracket (38) is the sum of three parts: the semidirect-product piece (38a) with generalized two-cocycle (38b) as in (8); and another semidirect-product piece (38c) for $\tilde{n}\,\mathbf{A}$ and $\tilde{n}$. Except for the generalized two cycle (38b), the bracket (38) is again associated to the dual of the Lie algebra L_2 in (26), but with a reinterpretation of dual coordinates. For SHMHD, on the second summand of (26) the dual coordinates are: $\tilde{n}\,\mathbf{A}$ dual to D, and $\tilde{n}$ dual to Λ^0. Hall magnetohydrodynamics for normal fluids is discussed from a Hamiltonian viewpoint in Holm.[18]

SUPERFLUID MAGNETOHYDRODYNAMIC (SMHD) APPROXIMATION OF MVSP

A magnetohydrodynamic approximation for the multivelocity superfluid plasma equations can be obtained from the SHMHD equations (35) by neglecting the Hall drift forces on the right-hand side of (34). The resulting system of SMHD equations becomes, with $\mathbf{B}=\mathrm{curl}\,\mathbf{A}$ in vector notation,

$$\partial_t\rho_\alpha = -\mathrm{div}(\mathbf{P}_\alpha + \rho_\alpha\mathbf{v}_n), \quad \alpha = 1,2 \tag{40a}$$

$$\partial_t S = -\mathrm{div}(S\mathbf{v}_n) , \tag{40b}$$

$$\partial_t\mathbf{v}^\alpha = -\nabla(\mu^\alpha - \tfrac{1}{2}v_n^2 + \mathbf{v}_n\cdot\mathbf{v}^\alpha) + \mathbf{v}_n\times\mathrm{curl}\mathbf{v}^\alpha , \tag{40c}$$

$$\partial_t P_i = -\partial_k\pi_i^k , \tag{40d}$$

$$\partial_t\mathbf{B} = \mathrm{curl}(\mathbf{v}_n\times\mathbf{B}) , \tag{40e}$$

where, now, the stress tensor in (40d) for SMHD is given by

$$\pi_i^k = \left[p + \frac{1}{2\mu_0} B^2 \right] \delta_i^k + (P_i v_n^k + P_\beta^k v_i^\beta - B_i B^k) \ . \tag{41}$$

The system of SMHD equations (40a)–(40e) preserves the total energy

$$H = \int d^3x \left[-\tfrac{1}{2}\rho v_n^2 + \mathbf{P}\cdot\mathbf{v}_n + \varepsilon + \frac{1}{4\mu_0} B_{ij} B_{ij} \right] , \tag{42}$$

with $\mathbf{P}=\mathbf{P}_1+\mathbf{P}_2+\rho\mathbf{v}_n$ and $\rho=\rho_1+\rho_2$. Consequently, this system is also a candidate for Hamiltonian formulation.

In fact, Eqs. (40a)–(40e) for SMHD do comprise a Hamiltonian system, with Hamiltonian (42) and the following Poisson bracket:

$$\begin{aligned}
\{H,F\} = -\int d^N x \Bigg[& \frac{\delta F}{\delta P_i} \left[(P_k \partial_i + \partial_k P_i) \frac{\delta H}{\delta P_k} + S \partial_i \frac{\delta H}{\delta S} + \rho_\beta \partial_i \frac{\delta H}{\delta \rho_\beta} + (\partial_k v_i^\beta - v_{k,i}^\beta) \frac{\delta H}{\delta v_k^\beta} \right] \\
& + \left[\frac{\delta F}{\delta S} \partial_k S + \frac{\delta F}{\delta \rho_\alpha} \partial_k \rho_\alpha + \frac{\delta F}{\delta v_i^\alpha} (v_k^\alpha \partial_i + v_{i,k}^\alpha) \right] \frac{\delta H}{\delta P_k} && \text{(43a)} \\
& + \left[\frac{\delta F}{\delta \rho_\alpha} \delta_\alpha^\beta \partial_k \frac{\delta H}{\delta v_k^\beta} + \frac{\delta F}{\delta v_i^\alpha} \delta_\beta^\alpha \partial_i \frac{\delta H}{\delta \rho_\beta} \right] && \text{(43b)} \\
& + \left[\frac{\delta F}{\delta P_i} (-B_{jl,i} + \partial_j B_{il} + \partial_l B_{ji}) \frac{\delta H}{\delta B_{jl}} + \frac{\delta F}{\delta B_{ml}} (B_{ml,k} + B_{kl} \partial_m + B_{mk} \partial_l) \frac{\delta H}{\delta P_k} \right] \Bigg] , && \text{(43c)}
\end{aligned}$$

associated with the dual space of the following Lie algebra:

$$L_3 = D \circledS [\Lambda^0 \oplus (\Lambda_\alpha^0 \underset{\alpha}{\oplus} \Lambda_\alpha^{N-1}) \oplus \Lambda^{N-2}] \ . \tag{44}$$

We see that L_3 for SMHD differs from L_1 in (9) for MVSF by the presence of an extra piece, Λ^{N-2}: the dual coordinates to this Λ^{N-2} are the $N(N-1)/2$ quantities B_{ij} in (43c); the two-cocycle piece in (43b) is the same as in (8b). In three dimensions, in terms of magnetic field $\mathbf{B}=\operatorname{curl}\mathbf{A}$, or $B^i=\varepsilon^{ijk}B_{jk}$ with ε^{ijk} totally antisymmetric, the piece (43c) of the Poisson bracket is expressible as

$$-\int d^3x \left[\frac{\delta F}{\delta P_i} (B^j \partial_i - \partial_k B^k \delta_i^j) \frac{\delta H}{\delta B^j} + \frac{\delta F}{\delta B^i} (\partial_k B^i - \delta_k^i B^j \partial_j) \frac{\delta H}{\delta P_k} \right] , \tag{43c$'$}$$

where the magnetic field B^i, $i=1,2,3$, being a flux (i.e., a 2-form) in three dimensions is dual to 1-forms Λ^1. The SMHD equations (40a)–(40e) are given in Hamiltonian form by $\partial_t F=\{H,F\}$ using Poisson bracket (43), with $F\in\{\rho_\alpha, S, \mathbf{v}^\alpha, \mathbf{P}, B_{ij}\}$ and H given in (42). This may be verified directly by substituting the variational derivatives of H obtained from

$$\begin{aligned}
\delta H = \int d^N x \Bigg[& \sum_\alpha (\mu^\alpha - \tfrac{1}{2} v_n^2) \delta\rho_\alpha + T\delta S + \sum_\alpha \mathbf{P}_\alpha \cdot \delta \mathbf{v}^\alpha \\
& + \mathbf{v}_n \cdot \delta \mathbf{P} + \left[\mathbf{P} - \sum \mathbf{P}_\alpha - \rho \mathbf{v}_n \right] \cdot \delta \mathbf{v} \\
& + (2\mu_0)^{-1} B_{ij} \delta B_{ij} \Bigg]
\end{aligned} \tag{45}$$

into the Poisson bracket (43). In three dimensions, the last term in the integrand of (45) becomes $\mathbf{B}\cdot\delta\mathbf{B}/\mu_0$ when expressed in terms of the magnetic field.

GENERALIZED KELVIN THEOREMS FOR SUPERFLUIDS

The interactions among the normal fluid and the superfluid condensates in MVSF, MVSP, SHMHD, and SMHD result in a generalized Kelvin theorem for each theory. These generalized Kelvin theorems show that the total circulation for each superfluid species is frozen into the *normal*-fluid motion. For example, in the case of MVSP, we obtain from (21c) and (21d) the following relation:

$$(\partial_t + L_{\mathbf{v}_n})(\mathbf{v}^\alpha + a^\alpha \mathbf{A})\cdot d\mathbf{x} = -d(\mu^\alpha - \tfrac{1}{2} v_n^2 - a^\alpha \mathbf{v}_n \cdot \mathbf{A}) , \tag{46}$$

where $L_{\mathbf{v}_n}$ denotes the Lie derivative with respect to the normal velocity $\mathbf{v}_n$. Hence, the circulation loop integral for each superfluid species,

$$\oint_{\gamma(t)} (\mathbf{v}^\alpha + a^\alpha \mathbf{A})\cdot d\mathbf{x} , \tag{47}$$

is conserved for every closed curve $\gamma(t)$ moving with the *normal* fluid. Let C^α be the superfluid circulation 1-form for species α

$$C^\alpha = \mathbf{C}^\alpha \cdot d\mathbf{x} = (\mathbf{v}^\alpha + a^\alpha \mathbf{A})\cdot d\mathbf{x} \ . \tag{48}$$

Then (46) is expressible more compactly as

$$(\partial_t + L_{\mathbf{v}_n}) C^\alpha = -dB^\alpha , \tag{49}$$

where B^α is the Bernoulli function on the right-hand side of (46). A brief calculation using (49) and the chain rule gives

$$(\partial_t + L_{\mathbf{v}_n})(C^\alpha \wedge dC^\alpha) = -d(B^\alpha dC^\alpha) \quad (\text{no sum on } \alpha) , \tag{50}$$

or, in components in three dimensions,

$$\partial_t(\mathbf{C}^\alpha \cdot \mathrm{curl}\, \mathbf{C}^\alpha) = -\mathrm{div}[\mathbf{v}_n(\mathbf{C}^\alpha \cdot \mathrm{curl}\, \mathbf{C}^\alpha) + B^\alpha \mathrm{curl} \mathbf{C}^\alpha] \quad (\text{no sum on } \alpha) . \tag{51}$$

Therefore, the *helicity* in three dimensions for each superfluid species, namely,

$$\lambda^\alpha = \int_D d^3x\, \mathbf{C}^\alpha \cdot \mathrm{curl}\, \mathbf{C}^\alpha \quad (\text{no sum on } \alpha) , \tag{52}$$

is conserved, provided the following boundary condition is satisfied:

$$[(\mathbf{C}^\alpha \cdot \mathrm{curl}\, \mathbf{C}^\alpha)\mathbf{v}_n + B^\alpha \mathrm{curl}\, \mathbf{C}^\alpha] \cdot \hat{\mathbf{n}} \mid_{\partial D} = 0 . \tag{53}$$

Subject to this boundary condition, the superfluid helicity λ^α is a *Casimir* for the MVSP Poisson bracket (20); that is, λ^α has vanishing Poisson brackets with *every* MVSP dynamical variable. Consequently, λ^α would be conserved for *any* Hamiltonian expressible in terms of these variables. The helicity λ^α in (52) represents the number of linkages of the vortex lines for species α in the domain of flow D. [See, e.g., Moffat[19] for further discussion of helicity in ordinary fluid mechanics and magnetohydrodynamics (MHD).]

The generalized Kelvin theorem (46) and conserved helicity (52) for MVSP reduces to the corresponding results for MVSF when $a^\alpha = 0$ for all α. Remarkably, the magnetohydrodynamic approximations SHMHD and SMHD also admit an additional conserved quantity, the magnetic helicity, in three dimensions

$$\lambda_{\mathrm{mag}} = \int_D d^3x\, \mathbf{A} \cdot \mathrm{curl}\, \mathbf{A} , \tag{54}$$

which is plainly a Casimir for the SHMHD Poisson bracket (38c). For SMHD, the Poisson bracket (43) can be expressed using the vector potential $\mathbf{A}$ as a dynamical variable by replacing (43c) by

$$-\int d^N x \left[\frac{\delta F}{\delta P_i}(\partial_j A_i - A_{j,i}) \frac{\delta H}{\delta A_j} + \frac{\delta F}{\delta A_i}(A_j \partial_i + A_{i,j}) \frac{\delta H}{\delta P_j} \right] , \tag{43c''}$$

for which A_i, $i = 1, 2, \ldots, N$, is dual to $(N-1)$-forms, Λ^{N-1}. Consequently, one finds by using (45) the vector-potential SMHD dynamics

$$\partial_t A_i = \{H, A_i\} \ [\text{by } (43''\text{c})] = -(\mathbf{v}_n \cdot \mathbf{A})_{,i} + (A_{j,i} - A_{i,j}) v_n^j . \tag{55}$$

Therefore, in three dimensions we have for SMHD

$$\partial_t(\mathbf{A} \cdot \mathbf{B}) = -\mathrm{div}[(\mathbf{A} \cdot \mathbf{B})\mathbf{v}_n] , \tag{56}$$

so that the magnetic helicity λ_{mag} in (54) is conserved for SMHD, as well, provided the normal fluid velocity $\mathbf{v}_n$ is tangential to the boundary. Note the similarity between the roles played by the superfluid velocities $\mathbf{v}^\alpha$ and the magnetic vector potential $\mathbf{A}$ in the Poisson brackets (43a) and (43c''), respectively. [Both $\mathbf{v}^\alpha$ and $\mathbf{A}$ are dual to $(N-1)$-forms.] In fact, using the three-dimensional curl of Eq. (40c) gives

$$\partial_t(\mathbf{v}^\alpha \cdot \mathrm{curl}\, \mathbf{v}^\alpha) = -\mathrm{div}[(\mathbf{v}^\alpha \cdot \mathrm{curl}\, \mathbf{v}^\alpha)\mathbf{v}_n + (\mu^\alpha - \tfrac{1}{2} v_n^2) \mathrm{curl}\, \mathbf{v}^\alpha] \quad (\text{no sum on } \alpha) . \tag{57}$$

Thus, SMHD also conserves the superfluid helicity

$$\lambda^\alpha = \int d^3x\, \mathbf{v}^\alpha \cdot \mathrm{curl}\, \mathbf{v}^\alpha \quad (\text{no sum on } \alpha) , \tag{58}$$

provided the term in square brackets in (57) is tangential to the boundary, [cf. (53)]. Subject to this boundary condition, the superfluid helicity λ^α in (58) is also a Casimir for both the SHMHD and SMHD Poisson brackets.

CONCLUSION

We have presented four nonlinear hydrodynamic theories describing nondissipative multicondensate solutions of superfluids, both charged and uncharged, and found the Hamiltonian structure for each theory. The multivelocity hydrodynamic equations for uncharged superfluid condensates in the density formulation (rather than the density *matrix* formulation proposed by Andreev and Bashkin[1]) are shown to possess a noncanonical Hamiltonian formulation. The Poisson bracket in this Hamiltonian formulation of MVSF is not symplectic: rather, it is associated to the dual space of the Lie algebra L_1 in (9) of semidirect-product type. The geometrical interpretations of the hydrodynamic superfluid variables as dual coordinates to this Lie algebra are physically natural: mass and entropy densities are dual to functions; superfluid circulation 1-forms are dual to $(N-1)$ forms in N dimensions; and the covector total momentum density is dual to vector fields. The induced drag interaction discovered in Andreev and Bashkin[1] and caused by relative motion among the superfluids arises in the present context via a generalized two-cocycle on the Poisson bracket. The total mass current density of the condensates then equals the total momentum density, i.e.,

$$\partial_t \rho = -\mathrm{div}\mathbf{P} \quad \text{with } \rho = \sum_\alpha \rho_\alpha \quad \text{and } \mathbf{P} = \rho \mathbf{v}_n + \sum \mathbf{P}_\alpha . \tag{59}$$

The superfluid circulation for each species in the MVSF theory is frozen into the *normal* fluid, as can be seen from the generalized Kelvin theorem for MVSF written in geometric form,

$$(\partial_t + L_{\mathbf{v}_n})(\mathbf{v}^\alpha \cdot d\mathbf{x}) = -d(\mu^\alpha - \tfrac{1}{2} v_n^2) , \tag{60}$$

which is simply Eq. (1d) in vector notation. Hence the superfluid vortex lines are frozen into the normal fluid, as seen by taking the exterior derivative of the generalized Kelvin theorem (60), or by taking the curl of (1d). In three-dimensional vector notation, this superfluid vorticity equation becomes

$$\partial_t(\mathrm{curl}\,\mathbf{v}^\alpha)=\mathrm{curl}(\mathbf{v}_n\times\mathrm{curl}\,\mathbf{v}^\alpha)\ . \tag{61}$$

Consequently, the superfluid helicity for each species in a finite domain of flow D, namely

$$\lambda^\alpha=\int_D d^3x\ \mathbf{v}^\alpha\cdot\mathrm{curl}\,\mathbf{v}^\alpha \quad (\text{no sum on } \alpha) \tag{62}$$

is readily shown to be conserved, provided the following boundary condition is satisfied:

$$[(\mathbf{v}^\alpha\cdot\mathrm{curl}\,\mathbf{v}^\alpha)\mathbf{v}_n+(\mu^\alpha-\tfrac{1}{2}v_n^2)\mathrm{curl}\,\mathbf{v}^\alpha]\cdot\hat{\mathbf{n}}\,|_{\partial D}=0 \quad (\text{no sum on } \alpha)\ . \tag{63}$$

The superfluid helicity λ^α in (62) is a Casimir for the MVSF Poisson bracket (8), in the sense that

$$\{\lambda^\alpha,G\}=0,\ \forall G(\mathbf{P},S,\rho_\alpha,\mathbf{v}^\alpha)\ . \tag{64}$$

That is, the helicity λ^α Poisson commutes with all of the MVSF dynamical variables and, so, would be conserved for *any* Hamiltonian expressible in these variables. The helicity λ^α may also be interpreted geometrically as a winding number, representing the number of linkages of vortex lines for superfluid species α in the three-dimensional domain of flow D.

The MVSF theory reduces to the standard equations of ideal adiabatic hydrodynamics for a normal fluid when $\mathbf{v}^\alpha=\mathbf{v}_n$ (so that all the fluids move together) and the generalized two-cocycle associated to (8b) in the Hamiltonian structure is absent. The Hamiltonian structure of MVSF is also given for the irrotational case in (8′). In the case of irrotational MVSF, the Hamiltonian structure is associated to the dual of the semidirect-product Lie algebra L_1' in (9′) with a *symplectic* two cycle between the condensate mass densities and corresponding superfluid velocity potentials.

Following this Hamiltonian pattern, we have generalized the MVSF theory to describe a multivelocity superfluid plasma (MVSP) in which some of the condensates are charged and interact electromagnetically among themselves and with an additional charged normal fluid (electrons). This generalization has been accomplished via the minimal-coupling hypotheses (14) and (16), along with alteration of the Hamiltonian to include the electronic and electromagnetic energies in (17). The MVSP theory extends the MVSF equations by modifying the motion equations for the charged condensates to include the Lorentz force and modifying the total momentum [see (21h)] and momentum flux tensor (22) to include contributions from the electrons and the electromagnetic field. Additional MVSP equations provide the hydrodynamics of the electrons and the Maxwell dynamics of the electromagnetic field. The electromagnetic coupling of the charged condensates with each other and with the normal electron fluid results in the drag-current density given in (24). The Hamiltonian structure of MVSP is associated to the dual space of the semidirect-product Lie algebra L_2 in (26) with two-cocycles associated to (20b) and (20d).

The Hamiltonian framework established here allows various extensions: to relativistic equations, for example, and to equations for superfluid plasmas either with internal spins and orbital angular momentum (see Holm and Kupershmidt[14]), or with Yang-Mills internal degrees of freedom (Gibbons, Holm, and Kupershmidt[20]). One can expect great richness of behavior from such systems combining the attributes of both superfluids and plasma dynamics.

Finally, we have developed two superfluid magnetohydrodynamic approximations of the MVSP theory: one approximation includes the Hall effect (SHMHD) and one neglects it (SMHD). In each case, the Hamiltonian structure for the approximate theory has been established and associated to the dual space of a semidirect-product Lie algebra. Remarkably, the two magnetohydrodynamic theories have essentially the same Hamiltonian, but have radically different Poisson brackets. For SHMHD, the superfluid contribution to the Hall drift force is identified explicitly in (34).

Each of the four superfluid theories discussed here is Hamiltonian and possesses an associated generalized Kelvin theorem, exemplified for MVSP in (46), and additional conservation laws—exemplified by the MVSP helicities λ^α in (52)—which are Casimir operators for the corresponding Poisson bracket (20).

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